A NOTE ON POLYNOMIAL
AND SEPARABLE GAMES

David Gale
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P-1216

Revised April 16, 1958

SUMMARY

It is shown that, given a pair of infinite metric spaces
and a pair of respective finite mixed strategies on them,
there exists a separable game with bounded continuous payoff
on their cartesian product such that the given strategies
constitute the unique solution of the game. If the spaces are
identical, then, corresponding to any given finite mixture,
one can find a symmetric polynomial-like game with bounded
(skew-symmetric) continuous payoff such that the given strategy
is the only optimal one. A stronger conclusion holds if the
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payoff can be a polynomial and have the desired property.

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1. INTRODUCTION

A two–person zero–sum game Γ is called polynomial–like or separable if its payoff function is of the form

$$M(x,y) = \sum_{i=1}^{n} f_i(x)g_i(y),$$

where x and y are elements of any strategy sets X and Y. Important special cases of separable games are those in which X and Y are bounded (usually compact) subsets of Euclidean spaces and M is a polynomial in the coordinates of x and y. These latter are called polynomial games.

It is a basic and fairly elementary fact concerning separable games [1], that, if optimal strategies exist, then these can always be chosen to be finite mixed strategies. We consider here the inverse question: Given a pair of finite mixed strategies, does there exist a separable (respectively, polynomial) game whose unique optimal strategies are the given pair? In case either X or Y is finite the answer is known to be in the negative. We here show, however, that

Theorem 1. If X and Y are metric spaces containing infinitely many points and μ and ν are any finite mixed strategies on X and Y respectively, then there is a payoff M, bounded continuous and separable on $X \times Y$, such that the associated game has μ and ν as unique optimal

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strategies.

Corollary. If X is a metric space containing infinitely many points and μ is any finite mixed strategy on X, then there is a skew–symmetric payoff M, bounded continuous and separable on X × X such that the associated symmetric game has .. as the unique optimal strategy.

For the case of polynomial games we show:

Theorem 2. If X and Y are bounded subsets of Euclidean spaces whose closures contain infinitely many points, then for any finite mixed strategies . and ν there exists a polynomial payoff function M such that the associated game has μ and ν as its unique optimal strategies.

(An analogous corollary holds here, also.)

Concerning the foregoing Theorem 2, we remark that Glicksberg and Gross, [2], have shown that any pair of mixed strategies can be the unique solution of a continuous game on the unit square. For finite mixtures, however, their construction is complicated, involving consideration of four special cases, and the payoff function is not a polynomial, nor even separable. The rather simple construction involved in our proof of Theorem 2 shows that their result still holds under the much stronger requirement that the payoff be a polynomial.

Finally, we credit Dresher, Karlin and Shapley, [1], for their rather exhaustive study of the structure of solutions of separable and polynomial games. However, their results do not include the theorems proved in this note. Indeed, one of the above authors has pointed out that the construction of the next

section provided a counter-example to one of the conclusions of
a structure theorem in |1|, and fortunately (for mathematics) an
error in the proof of that part of the theorem* was subsequently
uncovered.

2. POLYNOMIAL GAMES WITH PRESCRIBED UNIQUE SOLUTIONS

This section contains the proof of Theorem 2. Let X
and Y be sets satisfying the hypothesis of the theorem (We
pause to note that boundedness of X and Y is required to
insure integrability, since polynomials may otherwise be
unbounded.). Let μ be the mixed strategy which assigns the
weight μ_i to the point x_i of X, $i = 1, \ldots, m$, where
$\sum \mu_i = 1$. Similarly, let ν assign the weight ν_j to the
point y_j in Y, $j = 1, \ldots, n$ where $\sum \nu_j = 1$.

The set of points $\{x_1, \ldots, x_m\}$, the spectrum of μ,
will be denoted by $\sigma(\mu)$. Similarly, $\sigma(\nu)$ will denote the
spectrum of ν.

We now define the following set of polynomials:

$$f_0(x) = \prod_{x' \in \sigma(\mu)} |x - x'|^2,$$

$$f_i(x) = \prod_{x' \in \sigma(\mu) - \{x_i\}} \frac{|x - x'|^2}{|x_i - x'|^2}, \quad i = 1, \ldots, m,$$

where $|x - x'|$ is the usual Euclidean distance from x to
x'.

It is clear that the above functions are polynomials; how—

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*Theorem 7, fourth inequality pp. 175-176 of [1].

ever, aside from continuity, the only properties of them which we shall use are the following:

$$f_0(x) \geq 0 \quad \text{for all} \quad x \in X \quad \text{and} \quad f_0(x) = 0$$

if and only if $x \in \sigma(\mu)$.

$$f_1(x) \geq 0 \quad \text{for all} \quad x \in X \quad \text{and} \quad f_1(x) = 0$$

if and only if $x \in \sigma(\mu) - \{x_1\}$, $(1 = 1, \ldots, m)$.

$$f_1(x_1) = 1, \quad 1 = 1, \ldots, m.$$

In a precisely analogous manner we define the polynomials g_0 and g_j, $j = 1, \ldots, n$, on the set Y.

Next, let $\alpha_0, \alpha_1, \ldots, \alpha_n$ be $n + 1$ distinct cluster points of $\bar{X}$ (the closure of X) which do not meet $\sigma(\mu)$ (these exist by hypothesis), and define polynomials ϕ and ϕ_j, $j = 0, \ldots, n$ on X via

$$\phi(x) = \prod_{k=0}^{n} |x - \alpha_k|^2,$$

$$\phi_j(x) = \prod_{\substack{k \neq j \\ 0 \leq k \leq n}} |x - \alpha_k|^2, \quad j = 0, \ldots, n.$$

The only properties of these functions we shall use are that they are all non—negative, that ϕ vanishes only on the α_k, and that ϕ_j vanishes only on α_k with $k \neq j$.

Finally, let $\beta_0, \ldots, \beta_n$ be $m + 1$ distinct cluster points of $\bar{Y}$ which do not meet $\sigma(\nu)$, and define polynomials

γ and γ_1 on Y analogous to the functions ϕ and ϕ_j above.

We now define the desired payoff M by

$$M(x,y) = f_0(x)\phi(x)\left(g_0(y)\phi_0(x) + \sum_{j=1}^{n}(g_j(y) - \nu_j)\phi_j(x)\right)$$

(1)
$$- g_0(y)\gamma(y)\left(f_0(x)\gamma_0(y) + \sum_{i=1}^{m}(f_i(x) - \mu_i)\gamma_i(y)\right)$$

$$- \left(f_0(x)\phi(x)\right)^2 + \left(g_0(y)\gamma(y)\right)^2.$$

We show first that μ and ν are optimal strategies. If we compute $M(x,\nu)$ (in the usual extension), we obtain

(2)
$$M(x,\nu) = -\left(f_0(x)\phi(x)\right)^2 \leq 0.$$

To see this, it is sufficient to observe that, according to the properties noted above,

$$\int g_j d\tau = \nu_j$$

and g_0 and γ vanish on $\sigma(\nu)$. Similarly, we obtain

$$M(\mu,y) = \left(g_0(y)\gamma(y)\right)^2 \geq 0.$$

Thus μ and ν are optimal and 0 is the value of the game.

It follows also from (2) above that if μ' is any optimal strategy for player I, then the spectrum of μ' is contained in the zeros of $f_0\phi$. Thus any optimal μ' has weight only on the pure strategies x_1 and α_i, and similarly any optimal ν' for player II restricts its weight to $\{y_j\} \cup \{\beta_i\}$.

We now show that ν is the only optimal strategy for player II. For suppose ν' is optimal. Then, in the expression for $M(x, \nu')$, the second and fourth terms in (1) drop out in view of the remark of the preceding paragraph. Let ν' assign the weight ν_j' to the point y_j, and a weight ν_i'' to the point β_1. We show first that the numbers ν_i'' are all zero. For, let

$$(3) \qquad \gamma = \sum_{j=1}^{n} \nu_j' g_0(y_j) + \sum_{1=0}^{m} \nu_1'' g_0(\beta_1) = \sum_{1=0}^{m} \nu_1'' g_0(\beta_1).$$

Then, we have

$$(4) \qquad M(x, \nu') = f_0(x)\phi(x)\left[\gamma\phi_0(x) + \sum\lambda_j\phi_j(x) - f_0(x)\phi(x)\right];$$

where the λ_j are certain numbers, whose values will not affect the argument. Our assertion about ν_1'' will be proved if we can show that $\gamma = 0$, for then, from (3) and the properties of g_0, it will follow that each $\nu_1'' = 0$. Now, if $\gamma \neq 0$, then $\gamma > 0$. But recall that $\phi_0(\alpha_0) > 0$ and $\phi(\alpha_0) = \phi_j(\alpha_0) = 0$ for $j > 0$. But α_0 is a cluster point of $\overline{X}$; so there exists an $\overline{x} \in X$, such that at $x = \overline{x}$, the terms in the brackets of (4) get arbitrarily close to the positive number $\gamma\phi_0(\alpha_0)$, and if $\overline{x}$ is not one of the x_i or α_j then $f_0(\overline{x})\phi(\overline{x}) > 0$, so that we have $M(\overline{x}, \nu') > 0$, and ν' is not optimal.

Since all ν_1'' vanish, a simple computation shows that

$$(5) \qquad M(x, \nu') = f_0(x)\phi(x)\left[\sum_{j=1}^{n}(\nu_j' - \nu_j)\phi_j(x) - f_0(x)\phi(x)\right].$$

Finally, if $\nu' \neq \nu$, then, for some index k, $\nu'_k - \nu_k > 0$. But then, since a_k is a cluster point of $\overline{X}$, we can find $\overline{x}$ close to a_k but distinct from the x_i and a_j so that the bracketed term in (5) is close to the positive number $(\nu'_k - \nu_k)\phi(a_k)$ and so that $f_0(\overline{x})\phi(\overline{x}) > 0$. Thus, once again we would have $M(\overline{x}, \nu') > 0$ and ν' would not be optimal. The only remaining possibility is, therefore, that $\nu'_k = \nu_k$ for all k, so that $\nu' = \nu$ and uniqueness is established for player II.

Similarly, by a symmetric argument, $\mu' = \mu$ for any optimal strategy μ' for player I.

Thus, Theorem 2 is established.

3. METRIC SPACE GAMES—CONSTRUCTION OF PAYOFF

This section is dedicated to the construction of the payoff required for the establishment of Theorem 1 and its corollary, which will be proved in the final section. The construction and method of proof are quite similar to those used in proving Theorem 2; however, to preserve continuity of presentation, we shall paraphrase identical details.

Therefore, let X and Y be the respective spaces according to hypothesis, μ and ν the respective finite probability measures on them. μ and ν will be described with the same notations used previously. Finally, let ρ and ρ' denote the associated metrics of X and Y respectively. Then, without further ado, we initiate our construction.

The basis of our construction hinges on the fact that any

infinite metric space contains a sequence of disjoint neighbor-
hoods. To see this for X, say, there is no loss in generality
in assuming that X has a cluster point, for otherwise we are
guaranteed a sequence by the discrete topology induced by ρ
and the infiniteness of X. Therefore, let x^* denote a
cluster point of X. First, choose $\alpha_1 \neq x^*$, and, for $i > 1$
choose α_i so that $0 < \rho(x^*, \alpha_i) < \rho(x^*, \alpha_{i-1})/2$. Then, as
our sequence of neighborhoods, $\left\{ N_{\alpha_i} \right\}$, we set

$$N_{\alpha_i} = \left\{ x \mid \rho(x, \alpha_i) < r_i \right\}, \quad i = 1, 2, \ldots,$$

where $r_i = \rho(x^*, \alpha_i)/3$. It is easy to verify, using the
triangle inequality, that these neighborhoods are disjoint.

Therefore, let $\left\{ N_{\alpha_i} \right\}$ denote a sequence of disjoint
neighborhoods contained in X (spheres of radius r_i centered
at α_i). Define functions ϕ_j, $j = 0, \ldots, n$, as follows:

$$\phi_j(x) = \frac{r_i - \rho(x, \alpha_i)}{r_i} \cdot \frac{1}{i} \quad \text{if} \quad x \in N_{\alpha_i}$$

for some i (at most one) and
$i \equiv j \pmod{n + 1}$

$= 0 \quad \text{otherwise.}$

One verifies that ϕ_j is a bounded continuous function on X
into the non—negative reals, and which, moreover, satisfies

$$(6) \quad \begin{cases} \phi_j(\alpha_i) = \dfrac{1}{i} & \text{if} \quad i \equiv j \pmod{n + 1} \\ \quad = 0 & \text{otherwise.} \end{cases}$$

Next, let the function ϕ be given by

(7) $\qquad \phi(x) = \prod_{i=1}^{m} \rho(x, x_i), \quad x \in X,$

(where, as previously, $\{x_i\} = \sigma(\mu)$). There is no question about continuity here. We note merely that

(8) $\qquad \begin{cases} \phi(x) = 0 & \text{if } x \in \sigma(\mu) \\ \qquad > 0 & \text{otherwise.} \end{cases}$

Finally, we define functions f_j, $j = 0, \ldots, m$, as follows:

(9) $\qquad f_0(x) = \phi(x),$

and, for $j \in \{1, \ldots, m\}$, set

(10) $\qquad f_j(x) = \prod_{i \neq j} \dfrac{\rho(x, x_i)}{\rho(x_j, x_i)} .$

Here, again, continuity is immediate, and we note merely that

(11) $\qquad f_j(x_i) = \delta_{ij}, \quad i, j = 1, \ldots, m,$

where δ is Kronecker's delta. Moreover, to insure bounded-ness of these functions, if such is not the case, we need only replace ρ by the function $\dfrac{\rho}{1 + \rho}$ in the formulas (7) and (10) without affecting subsequent arguments.

The remainder of our construction involves defining certain bounded continuous functions on Y into the non-negative reals.

To accomplish this we merely repeat the foregoing construction with the replacements:

$$"X" \rightarrow "Y", \qquad "m" \rightarrow "n",$$
$$"\rho" \rightarrow "\rho'", \qquad "n" \rightarrow "m",$$
$$"x" \rightarrow "y", \qquad "\phi" \rightarrow "\psi",$$
$$"\alpha" \rightarrow "\beta", \qquad "f" \rightarrow "g",$$
$$"r" \rightarrow "r'", \qquad "\mu" \rightarrow "\nu".$$

In terms of these functions, then, and using the convention $\mu_0 = \nu_0 = 0$, we define our bounded continuous polynomial-like payoff M as follows:

$$M(x,y) = - \psi(y) \sum_{j=0}^{m} (f_j(x) - \mu_j)\psi_j(y)$$

$$+ \phi(x) \sum_{j=0}^{n} (g_j(y) - \nu_j)\phi_j(x)$$

$$- \phi(x) \sum_{j=0}^{n} \phi_j(x)^2 + \psi(y) \sum_{j=0}^{m} \psi_j(y)^2,$$

$(x,y) \in X \times Y$. This completes our construction.

4. VERIFICATION OF SOLUTION AND PROOF OF UNIQUENESS

To verify that (μ,ν) is a solution, we calculate first the expectation $M(\mu,y)$:

$$M(\mu,y) = \psi(y) \sum_{j=0}^{m} \psi_j(y)^2 \geq 0, \quad \text{all } y \in Y.$$

To see this, we note that the remaining sums vanish by virtue of (8), (9), and (11), i.e. ϕ vanishes on $\sigma(\mu)$ and

$\int f_j d\mu = \mu_j,$ $\quad j = 0, \ldots, m.$ $\quad$ Similarly,

(12) $\qquad M(x,\mathcal{v}) = -\phi(x) \sum_{j=0}^{n} \phi_j(x)^2 \leq 0,$ $\quad$ all $x \in X.$

Thus, $(\mu, \mathcal{v})$ is a solution and 0 is the value of the game.

$\qquad$ To show uniqueness for the first player, let μ' denote an optimal strategy for him. From the non-negativity of the functions ϕ, ϕ_j in (12), we see that $\int \phi \phi_j^2 d\mu' = 0$ for all $j \in \{0, \ldots, n\}$ and hence that $\int \phi \phi_j d\mu' = 0;$ for otherwise, by (12), a counter strategy is provided by $\mathcal{v}.$ Thus, if μ' is optimal, we have

(13)
$$M(\mu',y) = -\psi(y) \sum_{j=0}^{m} (\mu'_j - \mu_j) \psi_j(y)$$
$$+ \psi(y) \sum_{j=0}^{m} \psi_j(y)^2,$$

where we have written

$$\mu'_j = \int f_j d\mu', \quad j = 0, \ldots, m.$$

Next, suppose $\mu'_0 = \int f_0 d\mu' \neq 0$ (and hence, positive). Choose as possible counters a subsequence of the β's, $\left\{\beta_{n_1}\right\}$ such that $n_1 \equiv 0 \pmod{m+1}.$ Then, by virtue of the minimizer's counterpart of (6), (13) becomes

(14)
$$M(\mu',\beta_{n_1}) = -\psi(\beta_{n_1})\mu'_0 \cdot \frac{1}{n_1} + (\beta_{n_1})\frac{1}{n_1^2}$$
$$= \frac{\psi(\beta_{n_1})}{n_1}(-\mu'_0 + \frac{1}{n_1}), \quad 1 = 1, 2, \ldots .$$

Since ψ vanishes only on a finite set and is positive elsewhere, we see that the expression above can be made negative for i sufficiently large. Hence $\mu_0' = 0$, and it follows from (8) and (9) that $\sigma(\mu') \subseteq \sigma(\mu)$, i.e. any optimal μ' must restrict its spectrum to the set $\{x_1, \ldots, x_m\}$. Thus, finally, to establish uniqueness, we need only show that the corresponding weights are equal. Let μ_i' denote the weight on x_i placed by μ'. Substituting in our payoff M we obtain (noting $\mu_0' = 0$),

$$M(\mu',y) = -\psi(y) \sum_{j=1}^{m} (\mu_j' - \mu_j)\psi_j(y)$$

(15)

$$+ \psi(y) \sum_{j=0}^{m} \psi_j(y)^2.$$

Now suppose $\mu_k' \neq \mu_k$ for some $k \in \{1, \ldots, m\}$. Then, since $\sum_{j=1}^{m} \mu_j' = \sum_{j=1}^{m} \mu_j = 1$, we would have some $j = j_0 \in \{1, \ldots, m\}$ such that $\mu_{j_0}' > \mu_{j_0}$. But, by choosing the subsequence $\{\beta_{n_i}\}$ with $n_i \equiv j_0 \pmod{m+1}$, by the identical argument used before, we would find a counter rendering the expectation (15) negative. Hence $\mu_j' = \mu_j$ and thus $\mu' = \mu$. Uniqueness for the minimizer can be established in a similar manner, as is clear. So Theorem 1 is proved.

Finally, to establish the corollary, we need only make the appropriate identifications in our payoff to ensure that $M(x,y) = - M(y,x)$.

Added note.

The authors would like to thank Dr. Irving Glicksberg for

his valuable comments on this paper. As a matter of fact, Dr. Glicksberg suggested an alternate proof for Theorem 1 which extends it to completely regular spaces X, Y. The gist of his proof involves obtaining the extended theorem by making it a corollary of Theorem 1 via a mapping: $X \rightarrow R^n$, $Y \rightarrow R^m$.

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